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METHOD OF DERIVATION OF THE KORTEWEG – de VRIES – BURGERS EQUATION

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A method of derivation of the Korteweg – de Vries – Burgers (KdVB) equation for media with dispersion and dissociation, whose behavior is defined by equations of a fairly general form, is presented. The method is used for obtaining KdVB equations for collision plasma with Hall dispersion and the Korteweg – de Vries (KdV) equation for waves propagating in hot collisionfree plasma across a magnetic field.

Considerable attention was recently devoted to the investigation of the Korteweg – de Vries equation which provides a good definition of weakly nonlinear waves in the presence of dispersion in various media waves on shallow water, ionization sound in plasma, etc.). Since this equation is at present well known, its derivation is important for the investigation of wave motion in any medium. It was stated [1] on the basis of investigation of a number of examples that the

KdV equation is valid for wave motions in a certain medium only when solutions in the form of simple waves exist, when dispersion is disregarded, and the law of dispersion for small wave numbers is of the form $\omega = c_0 k - \beta k^3$, where ω is the frequency, k is the wave number, c_0 is the propagation velocity of oscillations in the absence of dispersion, and β is a dispersion parameter. Here proof is given of that statement for some limiting assumptions, and a general method of derivation of the Korteweg – de Vries – Burgers equation for such media. The KdV equation can be obtained from the latter by neglecting dispersion, and the Burgers equation by setting $\beta = 0$.

Note that a method similar to that presented here was used by A.G. Kulikovskii for the derivation of the Burgers equation.

1. Derivation of the KdVB equation in the case of existence of a complete system of eigenvectors. Let the state of some system be determined by the vector variable $\mathbf{u}$, which we represent in the form $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}'$, where $\mathbf{u}_0$ defines the stationary homogeneous state of the medium. The prime will be henceforth omitted. Let the behavior of the medium be defined by the system of equations of the form

$$\frac{\partial \mathbf{u}}{\partial t} + A\frac{\partial \mathbf{u}}{\partial x} = \mathbf{f}\left(\mathbf{u}_0 + \mathbf{u}, \ \frac{\partial \mathbf{u}}{\partial x}, \ldots, \frac{\partial^q \mathbf{u}}{\partial x^q}\right) \tag{1.1}$$

where $\mathbf{u}$ and $\mathbf{f}$ are vectors with components $u_1, \ldots, u_n$ and $f_1, \ldots, f_n$, respectively, and A is a matrix. We impose on $\mathbf{f}$ the following conditions:

1) if $\mathbf{u} \equiv 0$, then $\mathbf{f} \equiv 0$;

2) function $\mathbf{f}$ can be expanded into a Taylor series at least up to quadratic terms;

3) the term $\alpha\, \partial \mathbf{u}/\partial x$, where α is a constant matrix, is absent in the expansion of $\mathbf{f}$ into the Taylor series.

We call A the matrix of the system. In what follows we assume that all eigenvalues of matrix A are real and λ_1 is a single root. We impose the condition $(\lambda_i - \lambda_j)/\lambda_1 \sim 1$, when $\lambda_i \neq \lambda_j$. We shall consider only the case in which matrix A has n linearly-independent eigenvectors $\mathbf{C}_1, \ldots, \mathbf{C}_n$. The system of equations

$$\frac{\partial \mathbf{u}}{\partial t} + A\frac{\partial \mathbf{u}}{\partial x} = 0 \tag{1.2}$$

has then n linearly-independent solutions of the form $\varphi(t)\exp(ikx)$, which can be written as

$$\mathbf{u}_j = \mathbf{C}_j \exp[ik(x - \lambda_j t)], \quad j = 1, 2, \ldots, n$$

This solution determines waves which propagate at speeds $\lambda_1, \ldots, \lambda_n$.

Below we consider the first of these waves on the assumption that in the presence of dispersion and dissociation expressions for the frequency and attenuation increment of any wave are of the form

$$\omega_i = \lambda_i k - \beta_i k^3, \ \gamma_i = \nu_i k^2 \tag{1.3}$$
$$(\,|\beta_i| \lesssim |\beta_1|, \ \nu_i \geqslant 0, \ \nu_i \lesssim \nu_1)$$

Henceforth we shall denote λ_1 by c_0, and β_1 and ν_1 by β and ν, respectively.

We introduce new variables defined by formula

$$\mathbf{v} = C^{-1}\mathbf{u} \tag{1.4}$$

where vector $\mathbf{C}_i$ $(i = 1, \ldots, n)$ is the i-th column of matrix C. In these variables system (1.1) assumes the form

$$\frac{\partial \mathbf{v}}{\partial t} + B\frac{\partial \mathbf{v}}{\partial x} = \mathbf{g}\left(\mathbf{v}_0 + \mathbf{v}, \frac{\partial \mathbf{v}}{\partial x}, \ldots, \frac{\partial^q \mathbf{v}}{\partial x^q}\right)$$
$$B = C^{-1}AC = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$$
$$\mathbf{g}\left(\mathbf{v}_0 + \mathbf{v}, \frac{\partial \mathbf{v}}{\partial x}, \ldots, \frac{\partial^q \mathbf{v}}{\partial x^q}\right) = C^{-1}\mathbf{f}\left(C(\mathbf{v}_0 + \mathbf{v}), C\frac{\partial \mathbf{v}}{\partial x}, \ldots, C\frac{\partial^q \mathbf{v}}{\partial x^q}\right)$$

Let us introduce three small parameters: ε the ratio of perturbation amplitudes to related unperturbed parameters, the dispersion parameter $\delta = (\beta / c_0)^{1/2}L^{-1}$, where L is a characteristic length of the problem, and the attenuation parameter $\eta = \nu (c_0 L)^{-1}$. Subsequently we retain terms of the order of ε, ε^2, $\varepsilon\delta$, $\varepsilon\delta^2$ and $\varepsilon\eta$.

The subsequent analysis is entirely devoted to the Cauchy problem in an infinite space. We impose certain limitations on the initial conditions which imply that the considered wave propagates at velocity close to $\lambda_1 \equiv c_0$. Specifically we assume that $v_1 \sim \varepsilon$, $v_i = O(\varepsilon^2 + \varepsilon\delta + \varepsilon\eta)$, $i = 2, \ldots, n$ for $t = 0$.

In particular it is possible to set $v_i = 0$, $i = 2, \ldots, n$ for $t = 0$.

The equation for v_1 is written in the form

$$\frac{\partial v_1}{\partial t} + c_0\frac{\partial v_1}{\partial x} = g_1 \qquad (1.5)$$

Noting that $\mathbf{g} \equiv 0$ for $\mathbf{v} \equiv 0$, we have

$$g_1 = \sum_{i=1}^{n}\sum_{j=0}^{q} \alpha_{1ij}\frac{\partial^j v_i}{\partial x^j} + \sum_{i,j=1}^{n}\sum_{l,s=0}^{q} \beta_{1ijls}\frac{\partial^l v_i}{\partial x^l}\frac{\partial^s v_j}{\partial x^s} + o(\varepsilon^2) \qquad (1.6)$$

We introduce the dispersion and dissipation lengths defined, respectively, by formulas

$$l_1 = (\beta / c_0)^{1/2}, \ l_2 = \nu / c_0$$

From considerations of dimensionality we obtain

$$\alpha_{1i0} = c_0(a_{1i}l_1^{-1} + b_{1i}l_2^{-1}), \quad \alpha_{1ij} = c_0\sum_{r=0}^{j-1} l_1^r l_2^{j-r-1}\alpha_{1ij}^{(r)}, \quad (j > 0)$$
$$\beta_{1ij00} = c_0(a_{1ij}l_1^{-1} + b_{1ij}l_2^{-1})$$
$$\beta_{1ijls} = c_0 \sum_{r=0}^{l+s-1} l_1^r l_2^{l+s-r-1}\beta_{1ijls}^{(r)}, \quad l + s > 0$$

where a_{1i}, b_{1i}, $\alpha_{1ij}^{(r)}$, a_{1ij}, b_{1ij} and $b_{1ijls}^{(r)}$ are dimensionless quantities. The passing to limits $l_1 \to 0$ and $l_2 \to 0$ corresponds to the absence of dispersion and dissipation. Since for the considered system such passage is permissible $a_{1i} = b_{1i} = a_{1ij} = b_{1ij} = 0$ and, in virtue of assumption (3), $\alpha_{1i1} = 0$.

It is seen that the terms $\beta_{1ijls}^{(r)} l_1^r l_2^{l+s-r-1} (\partial^l v_i / \partial x^l)(\partial^s v_j / \partial x^s)$ are of the order $\delta^r \eta^{l+s-r-1} v_i (\partial v_j / \partial x)$, and, since for $i > 1$ we have $v_i = o(\varepsilon)$ hence it is possible to neglect in (1.6) all quadratic terms, except $(\beta_{11101} + \beta_{11110}) v_1 \partial v_1 / \partial x$. The order of the term of the form $l_1^r l_2^{j-r-1}\alpha_{1ij}^{(r)}\partial^j v_i / \partial x^j$ is $\delta^r \eta^{j-r-1} \partial v_i / \partial x$, therefore it is possible to neglect in (1.6) all linear terms, except

$$\alpha_{112}\frac{\partial^2 v_1}{\partial x^2}, \quad \alpha_{113}\frac{\partial^3 v_1}{\partial x^3}, \quad \alpha_{1i2}\frac{\partial^2 v_i}{\partial x^2}, \quad i=2,\ldots n$$

Thus, in the considered approximation we obtain for g_1 the formula

$$g_1 = \sum_{i=1}^{n} \xi_i \frac{\partial^2 v_i}{\partial x^2} + \zeta \frac{\partial^3 v_1}{\partial x^3} + \varkappa v_1 \frac{\partial v_1}{\partial x} \tag{1.7}$$

The equation for v_j, where $j = 2, \ldots, n$, is of the form

$$\frac{\partial v_j}{\partial t} + \lambda_j \frac{\partial v_j}{\partial x} = g_j \tag{1.8}$$

Since in (1.7) the term $\xi_j \partial^2 v_j / \partial x^2$ is of order $c_0 \delta \partial v_j / \partial x$, it is necessary to retain in the expansion of g_j into a Taylor series (1.7) only the term of order $c_0 \delta \partial v_1 / \partial x$, and neglect terms of higher order. In this manner (1.7) reduces to the equation

$$\frac{\partial v_j}{\partial t} + \lambda_j \frac{\partial v_j}{\partial x} = \theta_j \frac{\partial^2 v_1}{\partial x^2}, \quad \theta_j = \frac{\partial g_j}{\partial(\partial^2 v_1/\partial x^2)}\bigg|_{v=0} \tag{1.9}$$

We introduce the Fourier transform by formulas

$$v_{jk} = \int_{-\infty}^{\infty} v_j e^{-ikx} dx, \quad v_{jk\omega} = \int_{-\infty}^{\infty} v_{jk} e^{i\omega t} dt, \quad \operatorname{Im}\omega > 0$$

This reduces Eq. (1.9) to the form

$$i(k\lambda_j - \omega) v_{jk\omega} = -\theta_j k^2 v_{1k\omega} - \overset{\circ}{v}_{jk}, \quad \overset{\circ}{v}_{jk} = v_{jk}\big|_{t=0}$$

Applying the Fourier transform to Eq. (1.5), we obtain

$$i(kc_0 - \omega) v_{1k\omega} = -v_{1k}^{\circ} + O(\delta\varepsilon + \eta\varepsilon + \varepsilon^2)$$

The term $\theta_j k^2 v_{1k\omega}$ is of order $\delta\varepsilon$, consequently in the considered approximation we obtain

$$v_{jk\omega} = -\frac{\theta_j k^2 \overset{\circ}{v}_{1k}}{(kc_0 - \omega)(k\lambda_j - \omega)} + \frac{i\overset{\circ}{v}_{jk}}{k\lambda_j - \omega}$$

Using the inverse transformation, we obtain

$$v_j = -\frac{\theta_j}{c_0 - \lambda_j}\frac{\partial v_1^{\circ}}{\partial x}(x - c_0 t) + \frac{\theta_j}{c_0 - \lambda_j}\frac{\partial v_1^{\circ}}{\partial x}(x - \lambda_j t) + v_j^{\circ}(x - \lambda_j t) \tag{1.10}$$

where $v_j^{\circ} = v_j\big|_{t=0}$. We assume that

$$v_j^{\circ} = -\frac{\theta_j}{c_0 - \lambda_j}\frac{\partial v_1^{\circ}}{\partial x} \tag{1.11}$$

Hence in the considered approximation formula

$$\frac{\partial v_j}{\partial t} + c_0 \frac{\partial v_j}{\partial x} = 0 \tag{1.12}$$

is valid.

It follows from (1.9) and (1.12) that

$$\frac{\partial v_j}{\partial x} = \frac{\theta_j}{\lambda_j - c_0}\frac{\partial^2 v_1}{\partial x^2} \tag{1.13}$$

which reduces Eq. (1.5) to the form

$$\frac{\partial v_1}{\partial t}+c_0\frac{\partial v_1}{\partial x}=\varkappa v_1\frac{\partial v_1}{\partial x}+\xi_1\frac{\partial^2 v_1}{\partial x^2}+\left(\zeta+\sum_{j=2}^{n}\frac{\theta_j}{\lambda_j-c_0}\right)\frac{\partial^3 v_1}{\partial x^3} \tag{1.14}$$

Formulas (1.3) imply that relationships

$$\xi_1=\nu \quad \left(\zeta+\sum_{j=2}^{n}\frac{\xi_j\theta_j}{\lambda_j-c_0}\right)=-\beta \tag{1.15}$$

must be necessarily satisfied.

Setting $\varkappa\neq 0$ and using the substitution $w=-\varkappa v_1$, we obtain for w the KdVB equation

$$\frac{\partial w}{\partial t}+(c_0+w)\frac{\partial w}{\partial x}+\beta\frac{\partial^3 w}{\partial x^3}=\nu\frac{\partial^2 w}{\partial x^2} \tag{1.16}$$

It should be noted that condition (1.11) is unimportant.

In fact, if condition (1.11) is not satisfied, then in addition to the wave propagating at a velocity close to c_0 with amplitude of order ε waves running at velocities λ_j, with $j>1$ and amplitudes of order $\delta\varepsilon$ make their appearance. Because of this, additional terms of order $\delta^2\varepsilon$ which define the effect of these waves on the basic wave appear in the right-hand part of (1.14). However, the time of interaction between waves propagating at velocities λ_j with the basic wave, is of the order $t_1=L/(c_0-\lambda_j)\sim L/c_0$ for $c_0-\lambda_j\sim c_0$, and L is the characteristic width of the initial perturbation. The time during which these terms can materially affect v_1 may be estimated as $T=\delta^{-2}L/c_0$. Owing to this it is possible to neglect the terms related to the appearance of waves running at velocities λ_j, and Eq. (1.14) remains unchanged. All this is, however, valid only in problems with localized initial perturbation. In problems with periodic initial conditions, condition (1.11) is essential.

2. Derivation of KdVB equations in the case of absence of eigenvectors. Let us consider the case when the number of linearly independent eigenvectors of matrix A is smaller than n. For any matrix A there exists a matrix C such that $B=C^{-1}AC$ is a Jordan matrix. Let B have $M<h$ cells. We denote the dimension of the p cell by N_p. Owing to the imposed above condition of singleness of λ_1, we have $N_1=1$. Moreover, $\lambda_{S_p+1}=\ldots=\lambda_{S_p+N_p}=\lambda_p{}^*$, where $S_p=N_1+\ldots+N_{p-1}$ and $S_1=0$. Columns of matrix C are denoted by C_j as in Sect. 1. Note that now not all C_j are eigenvectors of matrix A.

The complete system of linearly-independent solutions of Eq. (2.2) of the form $\varphi(t)\exp(ikx)$ now becomes

$$\mathbf{u}_{S_p+j}=\sum_{r=1}^{j}\frac{(-ikt)^{j-r}}{(j-r)!}C_{S_p+r}\exp\left[ik\left(x-\lambda_p{}^*t\right)\right]$$
$$1\leqslant j\leqslant N_p,\quad p=1,\ldots,M$$

It will be seen that the oscillation modes to which correspond Jordan cells greater than unity are unstable owing to the appearance of secular terms. Because of this, all subsequent reasoning is valid only for fairly short time intervals during which instability had not yet developed. Note that the allowance for attenuation and dispersion in the analysis of the above mode may result in the disappearance of terms proportional to powers of t and, thus, eliminate the mode instability.

Proceeding as in Sect. 1 we find that the only difference is in that for the determination of v_j for $j > 1$ we have to consider the system of equations

$$\frac{\partial v_{S_{p}+1}}{\partial t} + \lambda_p{}^* \frac{\partial v_{S_p+1}}{\partial x} + \frac{\partial v_{S_p+2}}{\partial x} = g_{S_p+1}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{\partial v_{S_p+N_p-1}}{\partial t} + \lambda_p{}^* \frac{\partial v_{S_p+N_p-1}}{\partial x} + \frac{\partial v_{S_p+N_p}}{\partial x} = g_{S_p+N_p-1}$$

$$\frac{\partial v_{S_p+N_p}}{\partial t} + \lambda_p{}^* \frac{\partial v_{S_p+N_p}}{\partial x} = g_{S_p+N_p}$$

instead of Eq. (1.8).

We conclude, as in Sect. 1, that in the considered approximation it is possible to assume that $g_{S_p+j} = \theta_{S_p+j}\partial^2 v_1/\partial x^2$. Application of the Fourier transformation yields

$$v_{S_p+jk\omega} = -\frac{k^2 \overset{\circ}{v}_{1k}}{(kc_0-\omega)(k\lambda_p{}^*-\omega)} \sum_{l=0}^{N_p-j} \theta_{S_p+j+l}\left(\frac{k}{\omega-\lambda_p{}^*k}\right)^l -$$

$$\frac{i}{\omega-k\lambda_p{}^*} \sum_{l=0}^{N_p-j} \overset{\circ}{v}_{S_p+j+lk}\left(\frac{k}{\omega-\lambda_p{}^*k}\right)^l, \quad j=1,\ldots,N_p$$

with the specified accuracy.

Applying the inverse transformation, we obtain

$$v_{S_p+j} = -\sum_{l=0}^{N_p-j} \frac{\theta_{S_p+j+l}}{(c_0-\lambda_p{}^*)^{l+1}} \frac{\partial v_1{}^\circ}{\partial x}(x-c_0t) + \tag{2.1}$$

$$\sum_{l=0}^{N_p-j}\left[(-1)^l \frac{\theta_{S_p+j+l}}{c_0-\lambda_p{}^*} \frac{\partial^{l+1} v_1{}^\circ}{\partial x^{l+1}}(x-\lambda_p{}^*t) + \right.$$

$$\left.\frac{\partial^l v_{S_p+j+l}}{\partial x^l}(x-\lambda_p{}^*t)\right]\frac{t^l}{l!}$$

It will be readily seen that in the general case it is not possible to impose on initial conditions constraints which would reduce to zero the expression in brackets in the last equation (this could have been done, if the expression $(-1)^l\ \theta_{S_p+j+l}$ were independent of l when $l=0,\ \ldots,\ N_p-j;\ j=1,\ \ldots,\ N_p;\ p=1,\ \ldots,\ M$). However in a problem with localized perturbation propagation and fairly short times t, i.e. such for which the amplitudes of unstable modes remain small, it is possible to neglect the second sum in (2.1). Then, as in Sect. 1, v_{S_p+j} satisfies Eq. (1.12), and the reasoning in Sect.1 remains completely valid, except that instead of (1.15) we obtain

$$\zeta - \sum_{p=1}^{M}\sum_{j=1}^{N_p}\sum_{l=0}^{N_p-j} \frac{\theta_{S_p+j+l}}{(c_0-\lambda_p{}^*)^{l+1}} = -\beta$$

The system of equations of the form (1.1) with matrix A with multiple roots arises, for example, in the investigation of nonlinear waves propagating at an angle to the magnetic field in a cold plasma (see [1]). This is precisely the case in the considered problem, where zero is a double root.

3. The KdV equation for waves propagating across a magnetic field in a hot collisionless plasma. In the one-dimensional case the behavior of weakly-nonlinear waves in a magnetized collisionless plasma moving across a magnetic field whose directions remain constant, with the characteristic length of perturbations considerably greater than the Larmor radius, and the kinetic pressure of plasma greater than, or of the order of magnetic pressure, is defined by the following equations [2]

$$\frac{\partial U}{\partial t}+\frac{c_m{}^2}{\rho_0}=-U\frac{\partial U}{\partial x}+\Omega R^2\frac{\partial^2 V}{\partial x^2},\quad \Omega=\frac{eB_0}{cm_i},\quad R^2=\frac{p_{\perp 0}}{4\rho_0\Omega^2} \qquad (3.1)$$

$$\frac{\partial V}{\partial t}=-U\frac{\partial V}{\partial x}-\Omega R^2\frac{\partial^2 U}{\partial x^2},\quad \frac{\partial\rho}{\partial t}+\rho_0\frac{\partial U}{\partial x}=-\rho\frac{\partial U}{\partial x}-U\frac{\partial\rho}{\partial x}$$

$$c_m{}^2=(2p_{\perp 0}+B_0{}^2/4\pi)/\rho_0$$

where c_m is the propagation velocity of small perturbation in the absence of dispersion; Ω and R are, respectively, the Larmor frequency and ion radius; m_i is the mass of the ion; ρ_0, $\rho_{\perp 0}$ and B_0 are parameters of the unperturbed plasma. It will be seen that system (3.1) is of the form (1.1). The matrix of the system has three different eigenvalues: c_m, $-c_m$ and 0. Matrix C can be taken in the form

$$C=\begin{Vmatrix} c_m & c_m & 0\\ 0 & 0 & c_m\\ \rho_0 & -\rho_0 & 0\end{Vmatrix}$$

For g_1 we obtain (notation as in Sect. 1)

$$g_1=-\frac{3}{2}c_m v_1\frac{\partial v_1}{\partial x}+\frac{\Omega R^2}{2}\frac{\partial^2 v_3}{\partial x^2}$$

In the considered approximation $\partial v_3/\partial t=-\Omega R^2\partial^2 v_1/\partial x^2$. From this, using formulas (1.15), we obtain $\beta=-\Omega^2 R^4/2c_m$. In this way we obtain for v_1 the following equation:

$$\frac{\partial v_1}{\partial t}+c_m\left(1+\frac{3}{2}v_1\right)\frac{\partial v_1}{\partial x}-\frac{\Omega^2 R^4}{2c_m}\frac{\partial^3 v_1}{\partial x^3}=0 \qquad (3.2)$$

Since $U=c_m(v_1+v_2)$, where v_2 is a quantity of higher order than v_1, and in accordance with (1.12) in the considered approximation

$$\partial v_2/\partial t+c_m\partial v_2/\partial x=0$$

from (3.2) we obtain a similar equation for U.

4. The KdVB equation for waves in plasma with Hall dispersion and Joule dissipation. The system of equations which defines plasma with the Hall dispersion appears in [3]. It is shown in [4] that in spite of the presence of Joule dissipation the equation of entropy variation can in the considered approximation, be replaced by the condition $p=\rho^\gamma\cdot\mathrm{const}$. To allow for the Joule dissipation it is, thus, necessary to alter in the system of equations only the equation of induction, as is made in [4] (see Eq. (1)). Consequently the system of equations defining plasma with Hall dispersion and Joule dissipation, after linearization of dispersion and dissipation terms and rejection of terms of order higher than ε^2 is of the form

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} + \rho_0 \frac{\partial U}{\partial x} = -\rho \frac{\partial U}{\partial x} - U \frac{\partial \rho}{\partial x} \qquad (4.1)\\
&\frac{\partial U}{\partial t} + \frac{a_0^2}{\rho_0}\frac{\partial \rho}{\partial x} + \frac{B_0 \sin\alpha}{4\pi\rho_0}\frac{\partial b_z}{\partial x} = -U\frac{\partial U}{\partial x} - \frac{b_z}{4\pi\rho_0}\frac{\partial b_z}{\partial x} + \\
&\qquad \frac{B_0 \sin\alpha}{4\pi\rho_0}\rho\frac{\partial b_z}{\partial x} + \frac{a_0^2}{\rho_0}\rho\frac{\partial \rho}{\partial x} - \frac{a_0^2(\gamma-1)}{\rho_0^2}\rho\frac{\partial \rho}{\partial x} - \frac{b_y}{4\pi\rho_0}\frac{\partial b_y}{\partial x}\\
&\frac{\partial V}{\partial t} - \frac{B_0\cos\alpha}{4\pi\rho_0}\frac{\partial b_y}{\partial x} = -U\frac{\partial V}{\partial x} - \frac{B_0\cos\alpha}{4\pi\rho_0^2}\rho\frac{\partial b_y}{\partial x}\\
&\frac{\partial W}{\partial t} - \frac{B_0\cos\alpha}{4\pi\rho_0}\frac{\partial b_z}{\partial x} = -U\frac{\partial W}{\partial x} - \frac{B_0\cos\alpha}{4\pi\rho_0^2}\rho\frac{\partial b_z}{\partial x}\\
&\frac{\partial b_y}{\partial t} - B_0\cos\alpha\frac{\partial V}{\partial x} = -U\frac{\partial b_y}{\partial x} - b_y\frac{\partial U}{\partial x} + \frac{m_i c B_0}{4\pi e\rho_0}\cos\alpha\frac{\partial^2 b_z}{\partial x^2} \quad \nu_m\frac{\partial^2 b_y}{\partial x^2}\\
&\frac{\partial b_z}{\partial t} + B_0\sin\alpha\frac{\partial U}{\partial x} - B_0\cos\alpha\frac{\partial W}{\partial x} = \\
&\qquad -U\frac{\partial b_z}{\partial x} - b_z\frac{\partial U}{\partial x} - \frac{c m_i B_0}{4\pi e\rho_0}\cos\alpha\frac{\partial^2 b_y}{\partial x^2} + \nu_m\frac{\partial^2 b_z}{\partial x^2}, \quad a_0^2 = \frac{\gamma p_0}{\rho_0}
\end{aligned}
$$

where ρ_0 is the unperturbed density; ρ is the perturbation density; U, V and W are the x-, y- and z-components of the velocity vector; p_0 is the unperturbed pressure; B_0 is the induction of the unperturbed magnetic field (owing to the solenoidal properties of the magnetic field and induction, $b_x \equiv 0$); α is the angle between the vector of unperturbed magnetic field and the x-axis (in a system of coordinates in which that vector lies in xz-plane); and σ is the permeance. We assume that $\operatorname{tg}\alpha \sim 1$.

It will be readily seen that system (4.1) is of the form (1.1). The system matrix has six different eigenvalues

$$
\begin{aligned}
&\lambda_{1,2,3,4} = \pm a_{+,-} = \pm {}^1/_2 \left(\sqrt{a_0^2 + V_A^2 + 2a_0V_A\cos\alpha} \pm \right.\\
&\qquad \left.\sqrt{a_0^2 + V_A^2 - 2a_0V_A\cos\alpha}\right)\\
&\qquad\qquad \lambda_{5,6} = \pm V_A\cos\alpha, \qquad V^2{}_A = B^2{}_0/4\pi\rho_0
\end{aligned}
$$

Matrix C may be taken in the form

$$
C = \left\| \begin{matrix}
a_+ & -a_+ & a_- & -a_- & 0 & 0\\
0 & 0 & 0 & 0 & V_A\cos\alpha & -V_A\cos\alpha\\
-D_+ & D_+ & -D_- & D_- & 0 & 0\\
\rho_0 & \rho_0 & \rho_0 & \rho_0 & 0 & 0\\
0 & 0 & 0 & 0 & B_0 & B_0\\
Q_+ & Q_+ & Q_- & Q_- & 0 & 0
\end{matrix} \right\|
$$

$$
D_\pm = \frac{a_\pm^2 - a_0^2}{a_\pm}\operatorname{ctg}\alpha, \quad Q_\pm = \frac{a_\pm^2 - a_0^2}{V_A^2\sin\alpha}B_0
$$

The KdVB equation can be derived for a fast magneto-sonic wave ($c_0 = a_+$), as well as for a slow one ($c_0 = a_-$). In the first case we assume that at the initial instant $v_1 \sim \varepsilon$, $v_i = o(\varepsilon)(i = 2, 3, 4, 5, 6)$, and in the second $v_3 \sim \varepsilon$, $v_i = o(\varepsilon)(i = 1, 2, 4, 5, 6)$ (all notation as in Sect. 1).

For $g_{1,3}$ we obtain the following formula:

$$g_{1,3} = -A_{\pm}v_1\frac{\partial v_1}{\partial x} - \frac{V_A{}^2\chi\sin\alpha}{2(a_+^2 - a_-^2)}\frac{\partial^2}{\partial x^2}(v_5 + v_6) + \overset{*}{\nu}_{\pm}\frac{\partial^2 v_1}{\partial x^2}$$

where the plus and minus subscripts at A and $\nu_m{}^*$ relate to the fast and slow wave, respectively, and

$$A_{\pm} = a_+\left[1 + \frac{|a_0{}^2 - a_{\mp}^2|}{a_+{}^2 - a_-{}^2}\frac{\gamma - 2}{2} + \frac{|a_0{}^2 - a_{\mp}^2|(a_{\pm}^2 - a_0{}^2)^2}{2a_0{}^2(a_+{}^2 - a_-{}^2)V_A{}^2\sin^2\alpha}\right]$$

$$\overset{*}{\nu}_{m\pm} = \frac{\nu_m}{2}\frac{|a_{\pm}^2 - a_0{}^2|}{a_+{}^2 - a_-{}^2}, \qquad \chi = \frac{cm_iB_0\cos\alpha}{4\pi e\rho_0}$$

where the upper and lower subscripts relate to the fast and slow wave, respectively. Using formula (1.15) and noting that

$$\frac{\partial v_5}{\partial t} + V_A\cos\alpha\frac{\partial v_5}{\partial x} = \frac{\chi}{2}\frac{|a_{\pm}^2 - a_0{}^2|}{V_A{}^2\sin\alpha}\frac{\partial^2 v_p}{\partial x^2}$$

$$\frac{\partial v_6}{\partial t} - V_A\cos\alpha\frac{\partial v_6}{\partial x} = \frac{\chi}{2}\frac{|a_{\pm}^2 - a_0{}^2|}{V_A{}^2\sin\alpha}\frac{\partial^2 v_p}{\partial x^2}$$

where $p = 1$ for the fast wave, and $p = 3$ for the slow one, we obtain

$$\frac{\partial v_p}{\partial t} + a_{\pm}\frac{\partial v_p}{\partial x} + A_{\pm}v_p\frac{\partial v_p}{\partial x} + \beta_{\pm}\frac{\partial^3 v_p}{\partial x^3} = \overset{*}{\nu}_{m\pm}\frac{\partial^2 v_p}{\partial x^2}$$

$$\beta_{\pm} = -\frac{\chi^2 a_{\pm}|a_{\pm}^2 - a_0{}^2|}{2(a_+^2 - a_-^2)(a_{\pm}^2 - V_A{}^2\cos^2\alpha)}$$

Using again (1.15) and noting that b_z is a linear combination of v_i, for b_z we finally obtain

$$\frac{\partial b_z}{\partial t} + a_{\pm}\frac{\partial b_z}{\partial x} + \frac{V_A{}^2\sin\alpha}{a_{\pm}^2 - a_0{}^2}A_{\pm}\frac{b_z}{B_0}\frac{\partial b_z}{\partial x} + \beta_{\pm}\frac{\partial^3 b_z}{\partial x^3} = \overset{*}{\nu}_{m\pm}\frac{\partial^2 b_z}{\partial x^2} \qquad (4.2)$$

It should be noted that passing to new variables by formula (2.5) can be useful in the derivation of approximate equations for nonlinear waves also in the case, when the dispersion law differd from (2.3).

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